First and Second Order Vortex Dynamics

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The low energy dynamics of vortices in selfdual Abelian Higgs theory is of second order in vortex velocity and characterized by the moduli space metric. When Chern-Simons term with small coefficient is added to the theory, we show that a term linear in vortex velocity appears and can be consistently added to the second order expression. We provides an additional check of the first and second order terms by studying the angular momentum in the field theory. We briefly explore other first order term due to small background electric charge density and also the harmonic potential well for vortices given by the moment of inertia.

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I. INTRODUCTION

In the field theory solitons appear as nonperturbative classical objects which are interacting strongly each other when elementary quanta of the theory are interacting weakly. Among those solitons there exists a special class of the so-called selfdual solitons or BPS solitons. When they can be put together maintaining the BPS property, there is no static force between them and the solution space of multi-solitons is uniquely fixed by the moduli parameters. The description of their dynamics at low energy can be approximated as the moduli space dynamics. The metric of the moduli space is induced from the field theory and determines the exact form of the effective Lagrangian [1].

This idea has been successfully employed in the low energy dynamics of BPS magnetic monopoles in Yang-Mills Higgs theories [2] and selfdual vortices in Abelian Higgs theories [3]. (For the slow motion of selfdual vortices on Poincare upper half plane, there exists a beautiful work by Strachan [4].) There has been an attempt to find a similar effective Lagrangian for vortices in pure Chern-Simons Higgs theories [5]. While the theory is intrinsically relativistic, the low energy effective Lagrangian found in Ref. [5] was only of first order in vortex velocity. This first order effective Lagrangian was shown to be consistent and describes fractional statistics of vortices. However the attempt to find the second order expression was not successful. (See Ref [6] for other attempts.)

In this paper we start with the selfdual Abelian Higgs system whose low energy vortex dynamics is well known. Then we add the Chern-Simons term with small coefficient as a perturbation to the original theory and derive the correction to the vortex dynamics, which is the first order in both vortex velocity and the Chern-Simons coefficient. The first order expression can then be in the same order as in the second order one. By comparing the angular momentum from both field theory and low energy Lagrangian, we show that the low energy Lagrangian is consistent. By using this effective Lagrangian we explore two vortex dynamics in some detail. In addition we study briefly the first order term induced by the small uniform background charge density and also a small harmonic potential well for vortices given by the quadratic norm of the Killing vector for the angular momentum, which is the moment of inertia for rotation.

Our progress is inspired by a recent work in understanding the low energy dynamics of 1/2 BPS monopoles in N=4 supersymmetric Yang-Mills theories, when additional Higgs field takes nonzero expectation value [7]. The low energy Lagrangian of these monopoles consists of the kinetic and the potential parts. The 1/4 BPS or non BPS dyons arise in this effective Lagrangian naturally. The effective Lagrangian is valid in the regime where both kinetic and potential energies are much smaller than monopole mass but of the same order.

Vortices in Abelian Higgs theory was studied in early seventies [8] and shown to be selfdual when the Higgs coupling has the critical value [9]. The self-duality can be found even in the Chern-Simons-Higgs theories [10] and the mixed Maxwell-Chern-Simons theories [11]. Vortices in these theories carry fractional spin and satisfy fractional statistics [12,5]. Thus the low energy dynamics of these vortices seems to contain both second order and first order expressions in velocity.

The low energy effective Lagrangian for vortices in Abelian Higgs theories is of second order in velocity v and was derived by Samols [3]. When the Chern-Simons term with small coefficient κ is added, we expect that the low energy Lagrangian should be made of terms of order v^2 and terms of κv. Both of these terms are small and seem to be in the same order. One goal is to derive terms of order κv in the low energy Lagrangian.

Another goal is to show the consistency of the low energy Lagrangian . We calculate the conserved angular momentum from the field theory and express it in terms of the moduli space coordinates and velocities. We show that it is identical to what one gets from the low energy Lagrangian directly. This is true for both terms of order v^2 and of order κv.

We should note that the statistics of vortices with nonzero fractional spin is not given by the naive Aharonov-Bohm phase, rather it is given by the sum of the Aharonov-Bohm phase and the additional Berry phase due to the 'Magnus force' [5]. This has led to the explanation for the famous sign flip of vortex spin compared with the spin of elementary charged particles in the symmetric phase. The sign flip can be also seen from the transformation $\kappa \to -4\pi/\kappa$ of the Chern-Simons coefficient in the mirror map which makes vortices as elementary charged particles. Our first order Lagrangian again captures this interaction between vortices as well.

From the study of two vortex dynamics, we can expect nontrivial bound states with binding energy of order of elementary particle mass when the Chern-Simons coefficient κ is much larger than one [5]. The binding energy would be order of elementary particle mass. When κ has near zero, we do not expect any quantum bound state. This is consistent with our low energy Lagrangian which makes sense only for small κ.

One can ask whether or not there can be other perturbation of the vortex dynamics by adding other interactions to the theory. One case is to add a small uniform background electric charge, while keeping the selfduality. The first vortex dynamics in this theory is already done by one of us (K.L.) [13]. We can now add this to the quadratic Lagrangian when the background charge density ρ_e is very small in appropriate unit. This leads to an additional correction of order $\rho_e v$ to the low energy Lagrangian. We just need to translate the result of Ref. [13] to the present notation. In that work the angular momentum in the field theory is shown to be identical to that of the low energy effective Lagrangian of vortices.

More recently selfdual vortices in Abelian Higgs model with a first order kinetic energy has been studied [14]. The first order effective Lagrangian for slowly moving vortices has been obtained in this system and its angular momentum is shown to be consistent with that of the field theory. In this theory the Galilean symmetry is preserved and vortices do not carry any intrinsic spin.

One could also add some potential to the vortex dynamics. When the coupling constant in front of the scalar potential deviates a little from the critical value, there exists an induced potential between vortices [15]. Unfortunately this induced potential cannot be expressed in terms of vortex moduli coordinate explicitly unlike the kinetic term, making the analysis complicated. One may ask whether there exists other type of potential to add to the vortex dynamics. Inspired by the previously mentioned potential term to the magnetic monopole dynamics [7,16], we add a potential which is a quadratic norm of the angular momentum Killing vector, that is, the moment of inertia, arriving at a harmonic potential well for vortices.

The plan of this paper is as follows. In Sec. 2, we study the selfdual Abelian Higgs theory with additional Chern-Simons term. In Sec. 3, we rederive Samols' result for pure Abelian Higgs theory in somewhat different gauge. We also show that the angular momentum obtained from the field theory is identical to that from the Samols' second order Lagrangian. In Sec. 4, with the Chern-Simons term with small coefficient, we derive the first order term in the low energy effective Lagrangian. The angular momentum from the field theory is again shown to be identical to that from the first order term up to the constant term. The constant term is related to the intrinsic vortex spin. In Sec. 5, we study the two vortex interaction in detail. The anyonic nature of vortices is clearer here. In Sec. 6, we conclude with some remarks. In Appendix A we summarize the effect of the uniform background charge of Ref. [13]. In Appendix B we explore briefly the harmonic potential well given by the quadratic norm of the Killing vector of the angular momentum.

II. MODEL

The model we consider is a theory of a complex scalar field $\phi = fe^{-i\theta}$ coupled to the gauge field A_μ with the covariant derivative $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$. The Lagrangian density for this model [11] is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2}\partial_\mu N\partial^\mu N \\ &+\frac{1}{2}\partial_\mu f\partial^\mu f + \frac{1}{2}f^2(\partial_\mu\theta + A_\mu)^2 - U(N,f).\end{aligned} \tag{1}$$

The kinetic term for the gauge field has the usual Maxwell term and the parity violating Chern-Simons term. There is a neutral scalar field N which couples to the matter field. As we are interested in only the classical aspect of the theory, we put the electric coupling constant $e = 1$ for the convenience. We also scaled the fields and spacetime and so there is no dimensionful constant in the Lagrangian (1). The Gauss law constraint for the field configurations is

$$\nabla\cdot\mathbf{E} + f^2(\dot\theta + A_0) + \kappa B = 0, \tag{2}$$

where $E_i = F_{0i}$ and $B = F_{12}$.

There is a BPS bound on the energy when the interacting potential is chosen to be

$$U = \frac{1}{8}(f^2 - 1 - 2\kappa N)^2 + \frac{1}{2}f^2N^2. \tag{3}$$

To see this, let us reexpress the canonical energy density as

$$
\begin{aligned}
\mathcal{H} = & \frac{1}{2}(\mathbf{E} \pm \nabla N)^2 + \frac{1}{2}\left\{B \mp \frac{1}{2}(1 - f^2 + 2\kappa N)\right\}^2 + \frac{1}{2}(\dot{f}^2 + \dot{N}^2) \\
& + \frac{f^2}{2}(\dot{\theta} + A_0 \mp N)^2 + \frac{1}{2}\left\{\partial_i f \pm \epsilon_{ij} f(\partial_j \theta + A_j)\right\}^2 \pm \frac{1}{2}B \\
& \pm N\left\{\nabla \cdot \mathbf{E} + f^2(\dot{\theta} + A_0) + \kappa B\right\}.
\end{aligned} \tag{4}
$$

After the Gauss law is imposed, the energy $H = \int d^2x\mathcal{H}$ is bounded by the total magnetic flux $\Psi = \int d^2xB$,

$$
H \geq \frac{1}{2}|\Psi|. \tag{5}
$$

While the parity is broken by the Chern-Simons term, C and CTP are not broken and so both vortices and antivortices carry the same mass.

There are two possible vacua: the symmetric phase where $f = 0$ and $N = -1/(2\kappa)$ and the broken phase where $f = 1$ and $N = 0$. Currently we are interested in the symmetry broken vacua. In the broken phase there exist topological vortices. For n selfdual vortices the phase of the scalar field can be chosen to be

$$
\theta = -\mathrm{Im} \ln \phi = -\sum_{r=1}^{n} \mathrm{Arg}(\mathbf{r} - \mathbf{q}_r), \tag{6}
$$

where the position vectors of vortices are $\mathbf{q}_r$ with $r = 1, ..., n$. From the smoothness condition of the field ϕ at the position of the vortex, the moduli f of ϕ should have a simple zero at each $\mathbf{q}_r$. The finite energy condition implies that $\nabla\theta + \mathbf{A}$ vanishes quickly enough at the spatial infinity and so the total magnetic flux is quantized :

$$
\Psi = \int d^2xB = 2\pi n\ . \tag{7}
$$

Antivortices would have opposite winding and so carry negative magnetic flux.

The selfdual configurations are those saturating the energy bound (5). They satisfy some trivial equations

$$
\begin{aligned}
& E + \nabla N = 0, \quad \dot{f} = 0, \\
& \dot{N} = 0, \quad \dot{\theta} + A_0 - N = 0.
\end{aligned} \tag{8}
$$

In the gauge $\dot{\theta} = 0$, we get $A_0 = N$ and $\dot{A}_i = 0$, implying that the field configuration is static in time. In addition they satisfy the selfdual equations

$$
B = \frac{1}{2}(1 - f^2 + 2\kappa N), \tag{9}
$$

$$
\epsilon_{ij}\partial_i \ln f = (\partial_j\theta + A_j), \tag{10}
$$

$$
\nabla^2 N + f^2 N + \kappa B = 0, \tag{11}
$$

where the last one is from the Gauss law (2) with $A_0 = N$. The rest mass of the selfdual n vortices is then $n\pi$ and so each vortex carries mass π.

The angular momentum for the selfdual system can be obtained from the Noether procedure. The angular momentum with gauge invariant density is

$$J = -\int d^2x \epsilon_{ij} x^i \left\{ F_{0k}F_{jk} + \dot{N}\partial_j N + \dot{f}\partial_j f + f^2(\dot{\theta} + A_0)(\partial_j \theta + A_j) \right\}. \tag{12}$$

We will discuss this quantity for slowly moving vortices in the following sections.

III. THE ABELIAN HIGGS THEORY

For the well-known $\kappa = 0$ case, we review some aspects of selfdual vortices and their low energy effective Lagrangian obtained by Samols. This will provide the basic ground for the arguments in coming sections. In addition, we give a new nontrivial check of the Samols' result by showing the the field theoretic angular momentum is identical to one from the quadratic Lagrangian of Samols.

The selfdual equations (9) and (10) with $\kappa = 0$ can be put together into a single equation,

$$\nabla^2 \ln f^2 + 1 - f^2 = 4\pi \sum_r \delta^2(\mathbf{r} - \mathbf{q}_r). \tag{13}$$

With the asymptotic value $f = 1$, the solution of the above equation is uniquely determined by the set $\Gamma_n = \{\mathbf{q}_r; r = 1, ..., n\}$ [17]. Thus the moduli space of n selfdual vortices is defined by the positions of vortices. Since the configuration is invariant under the translation $\delta\mathbf{r} = \boldsymbol{\epsilon}$, and $\delta\mathbf{q}_r = \boldsymbol{\epsilon}$, we see that

$$\sum_r \frac{\partial}{\partial \mathbf{q}_r} \ln f^2 = -\frac{\partial}{\partial \mathbf{r}} \ln f^2. \tag{14}$$

Thus the center of the mass position $\mathbf{R} = (\sum_r \mathbf{q}_r)/n$ constitutes a complex plane C. The relative positions of vortices live on C^{n-1} modulo all permutations of the vortex positions. Calling the relative moduli space $\tilde{\mathcal{M}}_n \approx C^{n-1}/S_n$, where S_n is the permutation group of n objects. The total moduli space of n vortices will be

$$\mathcal{M}_n = C \times \tilde{\mathcal{M}}_n. \tag{15}$$

The energy density of n vortices approaches exponentially quickly to that of the vacuum outside vortex centers. As $\int d^2x(1 - f^2) = 2\pi n$ for n selfdual vortices, one can regard each vortex as a particle of area 2π and incompressible. While there is no repulsive force between vortices, n vortices occupy $2\pi n$ area.

It is convenient also to use the complex coordinates

$$z = x^1 + ix^2, \quad z_r = q_r^1 + iq_r^2 \ . \tag{16}$$

Note that $\partial_z = (\partial_1 - i\partial_2)/2$. By expanding around a vortex position z_r of the solution of Eq. (13), we get

$$\begin{aligned}\ln f^2 = \ln |z - z_r|^2 + a_r + \frac{1}{2}\left\{b_r(z - z_r) + \bar{b}_r(\bar{z} - \bar{z}_r)\right\} \\ +c_r(z - z_r)^2 + \bar{c}_r(\bar{z} - \bar{z}_r)^2 - \frac{1}{4}|z - z_r|^2 + \mathcal{O}(|z - z_r|^3).\end{aligned} \tag{17}$$

Since the field

$$\Phi = \ln \frac{f^2}{\prod_r |z - z_r|^2} \tag{18}$$

is nonsingular everywhere, we can see that

$$b_r = \sum_{s \neq r} \frac{2}{z_r - z_s} + \tilde{b}_r \tag{19}$$

with nonsingular function $\tilde{b}_r$. Note that b_r vanishes exponentially when vortices are separated from each other. The functions $b_r(z_s, \bar{z}_s)$'s will play the crucial role in the following argument.

A. Samols' Result

We are interested in the low energy dynamics of these selfdual vortices. As the moduli coordinates z_r's characterize the configuration uniquely and describe the zero modes, the time evolution of the field configuration will be approximated by the time evolution of the moduli coordinates $z_r(t)$. We choose the gauge where

$$\theta(\mathbf{r}, t) = -\sum_r \mathrm{Arg}(\mathbf{r} - \mathbf{q}_r(t)) \tag{20}$$

through the time evolution. (Samols has chosen the Weyl gauge $A_0 = 0$ and so his $\theta(t)$ is more complicated.)

To obtain the effective Lagrangian for the $z_r(t)$ variables, let us calculate the field theoretic Lagrangian for given initial data which are made of the field 'position' and 'velocity'. The field 'position' would be the self field configuration, $\mathbf{A}(\mathbf{r}; \mathbf{q}_r)$ and $\phi(\mathbf{r}; \mathbf{q}_r)$. The field 'velocity' is composed of E_i and $D_0\phi$ which could be regarded also as field 'momentum'. The time derivatives of the fields are given by $\dot{f} = \sum_r \dot{\mathbf{q}}_r \cdot \partial f/\partial \mathbf{q}_r$, and so on. Once we choose the above gauge (20), we cannot require $A_0 = 0$ anymore. The initial data should satisfy the Gauss law constraint which fixes the initial A_0 by

$$\partial_i E_i + f^2(\dot{\theta} + A_0) = 0. \tag{21}$$

This can put into a form

$$\begin{aligned} \partial_i \partial_0 (\partial_i \theta + A_i) - \partial_i^2 (\dot{\theta} + A_0) + f^2(\dot{\theta} + A_0) &= \partial_i [\partial_0, \partial_i] \theta \\ &= 2\pi \sum_r \dot{\mathbf{q}}_r \times \frac{\partial}{\partial \mathbf{q}_r} \delta^2(\mathbf{r} - \mathbf{q}_r). \end{aligned} \tag{22}$$

Due to the selfdual equation (10), the first term vanishes and the above equation becomes

$$\nabla^2 \mathrm{Im}\, \eta - f^2 \mathrm{Im}\, \eta = -2\pi \sum_r \dot{\mathbf{q}}_r \times \epsilon_{ij} \frac{\partial}{\partial \mathbf{q}_r} \delta^2(\mathbf{r} - \mathbf{q}_r), \tag{23}$$

where we have introduced a new field,

$$\eta = \frac{D_0 \phi}{\phi} = \partial_0 \ln f - i(\dot{\theta} + A_0). \tag{24}$$

From Eq. (13) for f, we can also get the similar equation for Re η. Together we get the equation for η,

$$4\partial_z \partial_{\bar{z}} \eta - f^2 \eta = 4\pi \sum_r \dot{z}_r \partial_z \delta^2(z - z_r), \tag{25}$$

where $\delta^2(z - z_r) = \delta^2(\mathbf{r} - \mathbf{q}_r)$. With the boundary condition $\eta = 0$ at spatial infinity, the unique solution of this equation (25) is

$$\eta = \sum_r \dot{z}_r \frac{\partial}{\partial z_r} \ln f^2. \tag{26}$$

Now we have the initial field data satisfying the Gauss law. Their field theoretic Lagrangian becomes

$$L = -n\pi + \frac{1}{2} \int_C d^2x \left\{ |E_1 + iE_2|^2 + |D_0 \phi|^2 \right\}, \tag{27}$$

where $n\pi$ is the rest mass of n vortices. As the initial field configuration is smooth everywhere, we can take out the positions of vortices from the integration region without changing the value of the integration. We call this region $\tilde{C} = C - \Gamma_n$. On this region $\tilde{C}$ we see that

$$E_1 + iE_2 = -2i \partial_{\bar{z}} \eta. \tag{28}$$

On C, the above relation does not hold as there are delta functions which do not vanish on Γ_n, but they are not in our integration region anymore. We will use this technique devised by Samols again and again with profitable results.

From the η field, we also get the initial field 'velocity', $D_0\phi$. Thus the low energy effective Lagrangian of order v^2 becomes

$$
\begin{aligned}
L_{v^2} = L + n\pi &= \frac{1}{2}\int_{\tilde{C}} d^2x \left\{4|\partial_{\bar z}\eta|^2 + f^2|\eta|^2\right\} \\
&= 2\int_{\Gamma_n} \partial_z(\bar\eta\partial_{\bar z}\eta) \\
&= -\sum_r \oint_{q_r} d\bar z\, \bar\eta\partial_{\bar z}\eta,
\end{aligned}
\tag{29}
$$

where we have used Eq. (25).

Near a vortex position z_r, the expansion (17) leads to

$$
\eta = -\frac{\dot z_r}{z - z_r} + \text{smooth terms}, \tag{30}
$$

and

$$
\partial_{\bar z}\eta = \frac{1}{4}\dot z_r + \sum_s \dot z_s \frac{\partial \bar b_r}{\partial z_s} + \mathcal{O}(|z - z_r|). \tag{31}
$$

After boundary integrations, we get

$$
L_{v^2} = \frac{\pi}{2} g_{rs}\dot z_r \dot{\bar z}_s, \tag{32}
$$

where the moduli space metric is

$$
g_{rs} = \delta_{rs} + 2\frac{\partial \bar b_s}{\partial z_r}. \tag{33}
$$

Since the L_{v^2} is real,

$$
\partial \bar b_s/\partial z_r = \partial b_r/\partial \bar z_s. \tag{34}
$$

When vortices are coming close to each other, the b_r's behave as in Eq. (19), and so the metric is regular. The kinetic energy is of order v^2 quantity and is correct when $v \ll 1$. The corrections of order v^4 are negligible in this nonrelativistic or low energy limit. The moduli space is Kähler as the two form

$$
w = \frac{i}{4} g_{rs} dz_r d\bar z_s \tag{35}
$$

is closed due to Eq. (34). Our complex coordinates z_r's are holomorphic coordinates with respect to this Kähler form.

The above effective Lagrangian has conserved angular momentum. We calculate it for the field theoretic initial 'data' of vortices moving slowly. Similar to the Lagrangian, we can express the field theoretic angular momentum in terms of moduli parameters, which turns out to agree with the one obtained from the low energy effective Lagrangian.

The field theoretic conserved angular momentum is

$$J = -\int_C d^2x\, \epsilon_{ij} x^i \left\{ F_{0k} F_{jk} + \dot{N} \partial_j N + \dot{f} \partial_j f + f^2 (\dot{\theta} + A_0)(\partial_j \theta + A_j) \right\}. \tag{36}$$

For the given initial data, the angular momentum becomes

$$J = -2i \int_{C-\Gamma_n} d^2x \left\{ z\partial_z (\bar{\eta} \partial_z \partial_{\bar{z}} \ln f^2) - \bar{z} \partial_{\bar{z}} \eta \partial_z \partial_{\bar{z}} \ln f^2 \right\}. \tag{37}$$

Now we note that, on $C - \Gamma_n$,

$$z\partial_z (\bar{\eta} \partial_z \partial_{\bar{z}} \ln f^2) = \partial_{\bar{z}} (z \partial_z \bar{\eta} \partial_z \ln f^2) \tag{38}$$

due to the equations (13) and (25) satisfied by f^2 and η. Thus, the integration can be converted to the boundary integrations,

$$J = \sum_r \left(\oint_{z_r} dz z\, \partial_z \bar{\eta}\, \partial_z \ln f^2 + c.c. \right). \tag{39}$$

From the expansions of $\ln f^2$ and η in Eqs. (17) and (30) around $z = z_r$, we get

$$J = \frac{\pi i}{2} g_{rs} (z_r \dot{\bar{z}}_s - \dot{z}_r \bar{z}_s) \tag{40}$$

which is exactly what we would get from the reduced Lagrangian (32). This is one more consistency check on the low energy effective Lagrangian (32).

Under the spatial translation $x^i \to x^i + a^i$, the angular momentum gets shifted by

$$J \to J + \epsilon_{ij} a^i P^j. \tag{41}$$

This is true both in the field theory and in the particle dynamics, and so does not provide any additional check. The field theoretic linear momentum is

$$P^i = -\int d^2x \left\{ \epsilon_{ij} E_j B + \dot{N} \partial_i N + \dot{f} \partial_i f + f^2 (\dot{\theta} + A_0)(\partial_i \theta + A_i) \right\}. \tag{42}$$

For the selfdual configurations, the complex momentum $P = P^1 + iP^2$ becomes

$$P = \int_{C-\Gamma_n} d^2z\, \partial_{\bar{z}} \left\{ \eta(1-f^2) \right\} \tag{43}$$

$$= \pi \sum_r \dot{z}_r \tag{44}$$

which is consistent with one from the Newtonian dynamics due to the translation invariance, $\sum_r \partial \bar{b}_s / \partial z_r = 0$.

IV. THE CHERN-SIMONS TERM

When $\kappa \neq 0$, we go back to the theory in Sec. 2. For the BPS configuration in the gauge (6), all fields are static in time and so $A_0 = N$. Especially the BPS equations (9)-(11) for vortices in the broken phase, $f = 1$ and $N = 0$, become two coupled equations

$$-\nabla^2 N + f^2 N + \frac{\kappa}{2}(1 - f^2 + 2\kappa N) = 0, \tag{45}$$

$$-\nabla^2 \ln f^2 + 1 - f^2 - 2\kappa N = 4\pi \sum_r \delta^2(z - z_r). \tag{46}$$

The zero modes of the vortex configuration are again given by the set of the positions of selfdual topological vortices [18]. The vortex configuration exists in the broken phase [19], and is expected to be unique for given vortex position. For $\kappa = 0$, the above equations become Eq. (13) for f^2, which studied in the previous section, and a homogeneous equation for N, which has a trivial solution $N = 0$. For small κ, the leading correction is N field of order κ. The higher order corrections to f^2 is of order κ^2 and that to N is of order κ^3. Here we consider only the leading order κ correction. The equation for the N field to order κ is

$$-\nabla^2 N + f^2 N + \frac{\kappa}{2}(1 - f^2) = 0, \tag{47}$$

where f^2 satisfies Eq. (13). We can find the closed form for N

$$\begin{aligned} N &= \frac{\kappa}{4} \left\{ \sum_r (\mathbf{r} - \mathbf{q}_a) \cdot \frac{\partial}{\partial \mathbf{q}_r} \ln f^2 \right\} \\ &= \frac{\kappa}{4} \left\{ -2n + \sum_r (\mathbf{r} - \mathbf{q}_a) \cdot \frac{\partial}{\partial \mathbf{q}_r} \ln \frac{f^2}{\prod_b |\mathbf{r} - \mathbf{q}_b|^2} \right\}. \end{aligned} \tag{48}$$

(This solution is inspired by the case in Ref. [13] where the Gauss law was solved similarly.) The N field is smooth everywhere and vanishes at spatial infinity.

A BPS configuration made of f and N fields is static but has nontrivial field 'momentum' of order κ as $E_i = -\partial_i N$ and $\dot{\theta} + A_0 = N$. Thus selfdual vortices carry nonzero electric charge

$$
\begin{aligned}
Q &= \int d^2x f^2(\dot{\theta} + A_0) \\
&= 2\pi n\kappa
\end{aligned}
\tag{49}
$$

which is exact in all order in κ. The static selfdual configuration also carries a nonzero angular momentum. For a rotationally symmetric configuration of n vorticity, the exact angular momentum is

$$
J_\kappa = -2\pi \int_0^\infty dr \frac{d}{dr} \left\{ rN'\bar{A} - \frac{\kappa}{2}\bar{A}^2 \right\} \tag{50}
$$

$$
= -\pi\kappa n^2, \tag{51}
$$

where we used $\partial_i\theta + A_i = \bar{A}(r)\hat{\varphi}^i/r$. Thus, we see that a single vortex carries charge $2\pi\kappa$ and spin $s = -\pi\kappa$.

When n vortices are on top of each other, the total angular momentum is $n^2 s$. On the other hand, the total angular momentum becomes just the sum ns when n vortices are in large separation. This looks strange but is expected from the spin statistics as we have explained in Ref. [5]. We review this briefly in the section where two vortex dynamics is discussed. We will argue that our effective Lagrangian is valid when κ is very small.

A. Effective Lagrangian

The low energy effective Lagrangian can be obtained again by evaluating the field theoretic Lagrangian for given field initial data which are made of the field 'position' and field 'velocity'. The leading order of the 'position' of the field should be given by f and N. For the BPS configurations, there exist still some nontrivial initial field momenta of order κ. We now want to include the field momenta of order v.

The initial field 'momentum' is divided into quantities of order κ and those of order v. The order κ terms are just from the N field. The additional 'momentum' would be proportional to v and given by the previous section. Basically, the initial 'momentum' of the vector field and scalar fields change from Eq. (28) and Eq. (24) to

$$
E_1 + iE_2 + (\partial_1 N + i\partial_2 N) = -2i\partial_{\bar{z}}\eta, \tag{52}
$$

$$
\partial_0 f - i(\dot{\theta} + A_0 - N) = \eta, \tag{53}
$$

on $C - \Gamma_n$ with η in Eq. (26).

We use this initial field data to evaluate the field theoretic Lagrangian. The leading zeroth order is the minus of the rest mass. The next order terms are made of terms of order $v^2, \kappa v$, and κ^2. We can see that order κ^2 should vanish as it represents just a shift of the rest mass or an additional potential, which is not there. After direct calculation of the Lagrangian, we get

$$L = -\pi n + L_{v^2} + L_{\kappa v1} + L_{\kappa v2}, \tag{54}$$

where

$$L_{v^2} = \frac{1}{2}\int d^2x \left\{ (\mathbf{E} - \nabla N)^2 + \dot{f}^2 + f^2(\dot{\theta} + A_0 - N)^2 \right\}, \tag{55}$$

$$L_{\kappa v1} = \kappa \int d^2x \left\{ \frac{1}{2}(\dot{A}_1 A_2 - \dot{A}_2 A_1) - B\dot{\theta} \right\}, \tag{56}$$

and

$$L_{\kappa v2} = \int d^2x \left\{ -(\mathbf{E} + \nabla N) \cdot \nabla N + f^2(\dot{\theta} + A_0 - N)N + \kappa B(\dot{\theta} + A_0 - N) \right\}. \tag{57}$$

The v^2 order Lagrangian L_{v^2} comes from the fact that vortices have the initial velocity. As there is no interference of order κ terms, L_{v^2} is identical to the Samols' result (29) discussed in the previous section.

Among the terms in L_{kv1}, it is easy to see the contribution from $A_2\dot{A}_1 - A_1\dot{A}_2$ vanishes due to the selfdual condition. The Lagrangian $L_{\kappa v1}$ is exactly identical with the Lagrangian we have obtained for the slow motion of vortices in the pure Chern-Simons Higgs theory in Ref. [5]. Even though the f field in that theory obeys a different equation, the character of the low energy Lagrangian is identical. While $\dot{\theta}$ is singular at each vortex position, it is not delta-function singularity. On $C - \Gamma_n$, we see that

$$\dot{\theta}B = \sum_r \dot{q}_r^i \partial_j \left\{ (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}) A_k \partial_l \ln |\mathbf{r} - \mathbf{q}_r| \right\}. \tag{58}$$

Thus

$$L_{\kappa v1} = 2\pi\kappa \sum_r \dot{\mathbf{q}}_r \cdot \mathbf{A}(\mathbf{r})|_{\mathbf{r}=\mathbf{q}_r}. \tag{59}$$

After inserting the asymptotic form for $\ln f^2$ in Eq. (17), we get the Lagrangian

$$L_{\kappa v1} = \frac{\pi\kappa}{2} \sum_r i(\dot{z}_r \tilde{b}_r - \dot{\bar{z}}_r \bar{\tilde{b}}_r). \tag{60}$$

This defines a natural one form Ω_1 on the moduli space

$$\Omega_1 = \frac{\pi\kappa}{2} \sum_r i(\tilde{b}_r dz_r - \bar{\tilde{b}}_r d\bar{z}_r). \tag{61}$$

From Eq. (19), we see that $\tilde{b}_r$ is smooth when vortices are coming together but has a long range tail when they are separated. This is what we need for the statistical phase interaction between vortex anyons.

One can impose the Gauss law on $L_{\kappa v2}$ by many different ways. The natural one is to replace f^2N by using Eq. (47) for N, which leads to

$$\begin{aligned}
&-(\mathbf{E}+\nabla N)\cdot\nabla N + f^2N(\dot\theta + A_0 - N) + \kappa B(\dot\theta + A_0 - N) \\
&= \partial_i\{\partial_i N(\dot\theta + A_0 - N)\} - \partial_i\{N\partial_0(\partial_i\theta + A_i)\}
\end{aligned} \tag{62}$$

on $C - \Gamma_n$. After contour integration, we get

$$L_{\kappa v2} = \pi\sum_r \dot{\mathbf{q}}_r \times \nabla N|_{\mathbf{r}=\mathbf{q}_r}. \tag{63}$$

We can find the asymptotic formula for the N field and so we get

$$L_{\kappa v2} = \frac{\pi\kappa}{8}\sum_r\left(i\dot z_r H_r + c.c.\right), \tag{64}$$

where

$$H_r = -b_r + \sum_{s\neq r}(z_r - z_s)\frac{\partial b_r}{\partial z_s} + \sum_{s\neq r}(\bar z_r - \bar z_s)\frac{\partial b_r}{\partial \bar z_s}\ . \tag{65}$$

This Lagrangian is not changed when we replace b_r by the regular $\tilde b_r$. Thus, this Lagrangian is regular everywhere and has no long range tail. This means that this Lagrangian does not provide any nontrivial statistics between vortices. This Lagrangian is purely local and affects only the short distance interaction between vortices. This Lagrangian introduces to the moduli space another one form,

$$\Omega_2 = \frac{\pi\kappa}{8}\sum_r\left(idz_r H_r + c.c.\right). \tag{66}$$

As there is no long range tail for this form, the field strength $d\Omega_2$ over any two dimensional plane will have zero net magnetic flux.

The sum of the Lagrangians in Eq. (60) and Eq. (64) forms the order κv effective Lagrangian $\mathcal{L}_{\kappa v}$ such as

$$\mathcal{L}_{\kappa v} = \mathcal{L}_{\kappa v1} + \mathcal{L}_{\kappa v2}. \tag{67}$$

When both κ and v are small and of the same order, it can be regarded as the same order Lagrangian $\mathcal{L}_v^2$ of Samols (32).

For the given initial field data, we can calculate the conserved angular momentum (36). The term $\dot{N}\partial_j N$ is of order $\kappa^2 v$ and can be neglected in the order we are working on. The angular momentum can be decomposed into terms of order κ and those of order v. The order v terms are

$$J_v = -\int d^2x\, \epsilon_{ij}x^i \left\{\epsilon_{jk}(E_k + \partial_j N)B + \dot{f}\partial_j f + f^2(\dot{\theta} + A_0 - N)(\partial_j\theta + A_j)\right\}. \tag{68}$$

For a given initial data, this expression is identical to Eq. (40) as expected.

The order κ correction to the angular momentum can be expressed

$$J_\kappa = -\int d^2x \epsilon_{ij}x^i \left\{f^2 N(\partial_j\theta + A_j) - \epsilon_{jk}\partial_k N B\right\}. \tag{69}$$

The order κ terms indicate the intrinsic angular momentum of the selfdual configuration. Using Eq. (11) for N, we can divide J_κ as a sum of

$$J_{\kappa 1} = \kappa \int d^2x\, \epsilon_{ij}x^i B(\partial_j\theta + A_j) \tag{70}$$

and

$$J_{\kappa 2} = -\int d^2x\, \epsilon_{ij}x^i \left\{(\partial_j\theta + A_j)\partial_k^2 N - \epsilon_{jk}\partial_k N B\right\}. \tag{71}$$

If we define the singular vector potential C_i as $C_i = \partial_i\theta + A_i$, the first angular momentum $J_{\kappa 1}$ is the expression we have obtained in Ref. [5] for the pure Chern-Simons Higgs theory. Employing the method in Ref. [5], we have

$$\begin{aligned} J_{\kappa 1} &= \kappa \int d^2x \partial_i \left(\frac{x^i}{2}C_j^2 - C_i x^j C_j\right) \\ &= -\pi\kappa n^2 + 2\pi\kappa \sum_r \mathbf{q}_r \times \mathbf{A}|_{\mathbf{r}=\mathbf{q}_r}. \end{aligned} \tag{72}$$

By the similar manipulation, the second angular momentum becomes

$$\begin{aligned} J_{\kappa 2} &= \int_{C-\Gamma_n} d^2x\, \partial_i \left(\epsilon_{jk}x^j \partial_i C_k - \epsilon_{jk}x^j N\partial_k C_i - \epsilon_{ij}NC_j\right) \\ &= -\pi \sum_r \mathbf{q}_r \cdot \nabla N|_{\mathbf{r}=\mathbf{q}_r}. \end{aligned} \tag{73}$$

We can see that this field theoretic angular momentum of order κ is identical to the angular momentum implied by the low energy effective Lagrangian of order κv given in Eq. (59) and Eq. (63) up to a constant term $-\pi\kappa n^2$ of Eq. (72).

V. TWO VORTEX DYNAMICS

A. Old Results

The moduli space $\mathcal{M}_2$ of two vortices can be split into R^2 for the center of mass motion and $\tilde{\mathcal{M}}_2$ for the relative motion. We introduce the center of mass position $Z = (z_1 + z_2)/2$ and the relative position $\zeta = (z_1 - z_2)/2$. In the center of the mass frame, there is obvious symmetry under the exchange of two vortices, which implies

$$b_2(\zeta) = -b_1(\zeta) = b_1(-\zeta). \tag{74}$$

The moduli space metric becomes

$$L_{\rm two} = \pi|\dot{Z}|^2 + \pi\left(1 + \frac{\partial \bar{b}_1}{\partial \zeta}\right)|\dot{\zeta}|^2. \tag{75}$$

As shown in Ref [3], in the polar coordinate $\zeta = \sigma e^{i\varphi}$, $b_1(\zeta) = b(\sigma)e^{-i\varphi}$ and so the metric for the relative motion becomes

$$L_{v^2} = \pi F^2(\sigma)(\dot{\sigma}^2 + \sigma^2\dot{\varphi}^2) \tag{76}$$

with $0 \le \varphi < \pi$ and

$$F^2(\sigma) = 1 + \frac{1}{\sigma}(\sigma b)' \ . \tag{77}$$

From Eq. (76) and the fact that the metric is smooth at the origin in terms of the well defined coordinate $\omega = \sigma^2 e^{2i\varphi}$, one can get the asymptotic expansion as follows:

$$b(\sigma) = \frac{1}{\sigma} - \frac{\sigma}{2} + c\sigma^3 + \mathcal{O}(\sigma^4) \tag{78}$$

near $\sigma = 0$ with nonzero positive c. For large σ, we get $b(\sigma) = \mathcal{O}(e^{-2\sigma})$. This leads to an integral formula $\int_0^\infty d\sigma\sigma(1 - F^2(\sigma)) = 1$. The Kähler potential for the metric is $K(\sigma) = \sigma^2/4 + \int^\sigma d\sigma' b(\sigma')$. The conserved angular momentum for the relative motion is

$$J_v = 2\pi F^2(\sigma)\sigma^2\dot{\varphi} \ . \tag{79}$$

The motion of vortices on moduli space is given by the geodesic. The relative moduli space of two vortices is a cone of deficit angle 180 degrees. Thus, the head-on collision of two vortices leads to 90 degree scattering [20]. The scattering of two vortices in the field theory was explored in detail numerically [21] and was found agreeing with the results from the effective Lagrangian [3].

B. First Order Term

As shown in the previous section, there are two kinds of contribution up to the first order terms in the low energy Lagrangian. The relative part of the first contribution for two vortices becomes

$$L_{\kappa v1} = -2\pi\kappa(\sigma b - 1)\dot{\varphi}\ . \tag{80}$$

This first order Lagrangian is smooth at $\sigma = 0$ and has a long range tail at the large distance. Its corresponding conserved angular momentum (72) becomes

$$\begin{aligned} J_{\kappa 1}(\sigma) &= -4\pi\kappa - 2\pi\kappa(\sigma b - 1) \\ &= -2\pi\kappa - 2\pi\kappa\sigma b \end{aligned} \tag{81}$$

which interpolates from $-4\pi\kappa$ at the origin to $-2\pi\kappa$ at large distance.

The second contribution becomes

$$L_{\kappa v2} = \frac{\pi\kappa}{2}\sigma(\sigma b)'\dot{\varphi}. \tag{82}$$

This first order Lagrangian is smooth at the origin and vanishes exponentially at infinity. There is no nontrivial long range interaction induced by this Lagrangian. The corresponding conserved angular momentum is

$$J_{\kappa 2} = \frac{\pi\kappa}{2}\sigma(\sigma b)'. \tag{83}$$

The low energy effective Lagrangian for two vortices in the center of mass frame is then

$$L_{\rm two} = \pi F^2(\sigma)(\dot{\sigma}^2 + \sigma^2\dot{\varphi}^2) + \mathcal{A}(\sigma)\dot{\varphi}, \tag{84}$$

where

$$\mathcal{A}(\sigma) = 2s(\sigma b - 1) - \frac{s}{2}\sigma(\sigma b)' \tag{85}$$

with vortex spin $s = -\kappa\pi$. The conserved Noether charge is

$$M = 2\pi\sigma^2 F^2\dot{\varphi} + \mathcal{A}, \tag{86}$$

while the total angular momentum is $J_{\rm tot} = 4s + M$. The conserved Hamiltonian is

$$\mathcal{H} = \frac{1}{4\pi F^2}\left\{ P_\sigma^2 + \frac{(M-\mathcal{A})^2}{\sigma^2} \right\}. \tag{87}$$

Note that $\mathcal{A}$ has a long range interaction even though it is smooth at the origin.

C. Quantum Mechanics

Let us start with recalling a general feature of two anyon dynamics in the center of mass frame. With fractional spin s, the orbital angular momentum is $2s+2l$ and so the total angular momentum is $4s+2l$. While anyon and antianyon carry the same sign spin s, their orbital angular momentum eigenvalue is $-2s+2l$ and so they can annihilate each other to the vacuum.

Anyons can be treated as bosons with electric charge and magnetic flux, and so have a long range Aharonov-Bohm like interaction. Equivalently, they can be treated as particles of fractional spin without any further long range gauge interaction. This can be done also for the quantum mechanics of two Chern-Simons vortices.

Let us consider the quantum mechanics of this system briefly. The Hamiltonian (87) for the relative motion in the operator becomes

$$\mathcal{H} = -\frac{1}{4\pi\sigma F^2}\partial_\sigma(\sigma\partial_\sigma) + \frac{1}{4\pi\sigma^2 F^2}(-i\partial_\varphi - \mathcal{A})^2. \tag{88}$$

After quantization, we treat vortices as identical bosons with a long range magnetic interaction. Then the wave function Φ_{boson} for the two bodies should be single-valued under exchange. As the range of φ is $[0,\pi]$, $\Phi_{\text{boson}} \sim e^{i2l\varphi}$ and the orbital angular momentum $M = -i\partial/\partial\varphi$ has the even integer eigenvalues $2l$'s. The wave function is well defined on the moduli space.

However, this is not the whole story. The eigenvalue of the conserved total angular momentum becomes

$$J_{\text{tot}} = 4s + M = 4s + 2l \tag{89}$$

of which $2s$ would be the sum of the vortex intrinsic spin. Thus, it is better to interpret the $2l+2s$ as the eigenvalue of the new orbital angular momentum $M_{orb} = M + 2s$. Then the anyonic wave function

$$\Phi_{\text{anyon}} = e^{2is\varphi}\Psi_{\text{boson}} \tag{90}$$

is the eigenstate of the $M_{\text{orb}} = -i\partial_\varphi$ with eigenvalue $2l+2s$ as expected for two anyons of spin s. The Hamiltonian for Ψ_{anyon} becomes

$$\mathcal{H}_{\rm anyon} = -\frac{1}{4\pi\sigma F^2}\partial_\sigma(\sigma\partial_\sigma) + \frac{1}{4\pi\sigma^2F^2}(-i\partial_\varphi - \mathcal{A}')^2 \tag{91}$$

with $\mathcal{A}' = 2s\sigma b - s\sigma(\sigma b)'/2$, which is smooth and provides a short range interaction. This is exactly what is expected for anyons.

Classically, the orbital angular momentum of two selfdual vortices changes from $2s$ to zero as their separation increases. For nonzero s, two overlapped vortices cannot escape to infinity with very small energy as argued in Ref. [5]. The reason is that their interaction is short ranged and the spatial motion cannot carry enough orbital angular momentum when the available kinetic energy is arbitrarily small as one needs a large impact parameter. Similarly, two vortices in large separation cannot come together with very low initial kinetic energy.

To see this quantum mechanically, we first note that the classical approximation is good when the classical quantity is much larger than $\hbar$. Thus the classical picture of vortex spin is fine if the vortex spin $s = -\pi\kappa$ is much larger than unity. Assuming the quantum correction is very small, say due to the underlying supersymmetry, one can see that the selfdual two vortex states, saturating the BPS energy, could have the orbital angular momentum $2s + 2l$ with the integer l which ranges from zero to $\approx -s$, depending on how close they are. From the classical picture, two vortices will be on top of each other when $M_{\rm orb} = 2s$ and will be far apart when $M_{\rm orb} = 0$. As the interaction is short ranged, the quantum states with $|M_{\rm orb}| = |2s + 2l| \geq 1$ would be bound states of vortices of zero energy with size of vortex core. The bound energy of these states without supersymmetry would be the order of elementary particle mass. The quantum state with $M_{\rm orb} = 2s + 2l \approx 0$ would be the ground state of states of the continuum energy.

We have argued that the moduli space approximation is good when $s = -\pi\kappa$ is very small. Quantum mechanically we do not expect any bound state and states with large bound energy. Thus, our approximation of the low energy dynamics of vortices in small v and κ limit with Lagrangians of order v^2 and κv is consistent.

VI. CONCLUSION

From the field theoretic Lagrangian, we have obtained the low energy effective Lagrangian for vortices in the selfdual Abelian Higgs theory. The gauge theory has both Maxwell and Chern-Simons kinetic terms. The effective Lagrangian consists of the quadratic and linear in small vortex velocity. We have argued that the valid regime of this effective Lagrangian is the case where Chern-Simons coefficient is small. We have also shown that our effective Lagrangian has the conserved angular momentum consistent with the field theoretic one. We have studied two vortex dynamics briefly.

In appendix we have shown the first order term induced by the small background charge and also a harmonic potential well given by the moment of inertia. We see that the general consistent dynamics of vortices contains second and first order terms and a potential. It would be very interesting to find the detailed dynamics of two vortices in this frame.

The supersymmetric generalization of our Lagrangian would also be very interesting. In supersymmetric theories [11] we expect that there may be many threshold bound states of vortices in large κ limit. Another intriguing question may be to see whether or not this is true in our low energy Lagrangian.

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Appendix A: Uniform Background Charge

We can add the uniform background electric charge density to the Lagrangian,

$$\mathcal{L}_{\rho_e} = -\rho_e A_0 \; . \tag{92}$$

The energy function still gets the BPS bound if we choose the potential to be

$$U = \frac{1}{8}(1 - f^2 + 2\kappa N)^2 + \frac{1}{2} f^2 N^2 - \rho_e N. \tag{93}$$

Interestingly the BPS bound appears with only one sign when one mixes the Chern-Simons terms and the background charge [22]. When both κ and ρ_e are very small, we expect the correction of order $\rho_e v$ and $\kappa\rho_e$. The vacuum structure of this theory is much more complicated and somewhat similar to those studied in Ref. [22] (See below in Table 1).

vacuum	$\kappa\rho_e$	f^2	N
broken	$\kappa\rho_e > -1/8$	$\frac{1+\sqrt{1+8\kappa\rho_e}}{2} \xrightarrow{\rho_e \to 0} 1$	$\frac{2\rho_e}{1+\sqrt{1+8\kappa\rho_e}} \xrightarrow{\rho_e \to 0} 0$
	$-1/8 < \kappa\rho_e \leq 0$	$\frac{1-\sqrt{1+8\kappa\rho_e}}{2} \xrightarrow{\rho_e \to 0} 0$	$\frac{2\rho_e}{1-\sqrt{1+8\kappa\rho_e}} \xrightarrow{\rho_e \to 0} -\frac{1}{2\kappa}$
	$\kappa\rho_e = -1/8$	$1/2$	$2\rho_e$
	$\kappa\rho_e < -1/8$	not real	not real
symmetric	all values	0	$\frac{1}{2\kappa}\left(\frac{2\rho_e}{\kappa} - 1\right) \xrightarrow{\rho_e \to 0} -\frac{1}{2\kappa}$

Table 1

We are interested in selfdual vortices in the broken phase $\langle f \rangle \approx 1, ; \langle N \rangle \simeq 0$ when the small κ and ρ_e limit is taken. For small κ and ρ_e limit, their contributions do not mix each other. As there is no correction to the mass of vortices, there is no $\kappa\rho_e$ order correction. The $\rho_e v$ correction in Ref. [13] can be written as

$$L_{\rho_e v} = \frac{\pi\rho_e}{2} \sum_r \{ i\dot{z}_r(\bar{z}_r - H_r) + cc \} \tag{94}$$

whose conserved angular momentum is

$$J_\rho = \pi\rho_e \sum_r \left\{ |z_r|^2 + (z_r H_r + cc) \right\} . \tag{95}$$

A single vortex feels the uniform magnetic field by the first order term. This is due to the combination of Lorentz and Magnus forces on the vortex. When the background charge is very small, the Landau level energy will be much smaller than the vortex mass, making the low energy effective action possible.

As there is BPS bound for only one sign in the energy density, one suspects that there may be a potential of order $\kappa\rho_e$ for the moduli space dynamics of antivortices. It remains to be clarified in future.

Appendix B: Harmonic Well Potential

When the coupling constant for the field theory potential is tiny bit different from the critical value, the induced potential on the vortices can be included in the low energy Lagrangian as shown in Ref. [15]. However, the explicit form of the vortex potential is not available and so the analysis is more complicated.

On the other hand, one can imagine a different type of potential on vortices. For example, we can imagine the magnetic flux on two plane is bundled by an external magnet in third space, like vortices on superconducting large disc lying between two poles of a magnet. It is not easy to incorporate such a potential in the field theory Lagrangian. However, it is easy to add a 'confining' potential on vortices.

The second order Lagrangian has an obvious rotational symmetry. We have discussed the corresponding conserved angular momentum extensively. On the moduli space with the metric by Samols, there exists a corresponding Killing vector

$$K = \sum_r i(z_r \partial_{z_r} - \bar{z}_r \partial_{\bar{z}_r}). \tag{96}$$

This is holomorphic Killing vector. In the discussion of 1/4 BPS magnetic monopole dynamics [22,16], the additional potential was given by the quadratic norm of the triholomorphic Killing vectors for the internal symmetry. Similarly, we find the quadratic norm of the angular momentum Killing vector as a potential

$$U_{\rm eff} = \frac{\lambda}{2} g_{rs} z_r \bar{z}_s \tag{97}$$

with a small positive coefficient λ. This potential leads to the harmonic potential well to the vortices.

The quadratic norm of the angular momentum Killing vector is of course the moment of inertia we know well. Thus the above potential is proportional to the inertia. The inertia can be split into one for the center of mass motion and another for the relative motion. Thus, we can have one potential for the center of mass motion and another potential for the relative motion with different coefficient.

When λ is of order v^2, we can include this in the low energy Lagrangian. When vortices are close to each other, the configuration of them is very complicated but roughly one can see that vortices are incompressible particles on two dimensions. The reason is that $\int d^2x(1-f^2)/(4\pi) = n$ measures the vortex core area and is increasing with vortex number. Our potential is a harmonic well and so the low energy dynamics of $L_{v^2} + U_{\rm eff}$ describes particles of finite size hard core and harmonic well. All interactions are short ranged. Of course one can add the first order Lagrangians too. As the angular momentum Killing vector is holomorphic, the supersymmetry can be extended to include the potential.

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